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Maximum leakage times through puncture holes for process vessels of various shapes

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Abstract

In this work, equations are developed for the drainage of vessels of different geometrical shapes through a side leak. From these equations, the maxima in the leakage times, divided by the time required to drain the full vessel from its bottom as a function of the side leak elevation, are determined. Five different geometric shapes are considered here — spheres, cylinders, cones, paraboloids and ellipsoids. The equations and maxima for the latter three configurations have not previously appeared in the literature.

1. Introduction

There has been a resurgence of interest in the classical subject of tank drainage — at least partially from safety considerations — in recent years [1]. Most of the early formulas developed to compute drainage (or efflux) times from process or storage vessels were developed for tanks draining through a hole at the bottom [2]. Some of these formulas are beginning to find their way into recent textbooks on process safety [3, 4].

Because of increasing concerns about safety and loss prevention, there has recently been developed the need for accurate formulas to compute fluid discharge and vessel emptying rates for an opening of a given size and at an arbitrary elevation. Such a need arises, for example, in analyzing an accident scenario resulting from a moving vehicle, e.g., a forklift truck or an autonomous guided vehicle (AGV), being driven into the side of a vessel. Thus, such formulas were recently presented by Crowl [5] and by Hart and Sommerfeld [6] for spheres and vertical cylindrical vessels: a numerical method for estimating such discharge rates and times was also developed by Crowl for horizontal cylindrical vessels. Sommerfeld and Stallybrass [7] developed an analytical

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formula, incorporating elliptic integrals for this latter geometry, and confirmed Crowl's numerical results.

In his work, Crowl [5] introduced a dimensionless drainage or efflux time (τ), defined as the time required to drain an initially completely full vessel down to the elevation of the opening for liquid flow divided by the time required for the same initially full vessel to drain completely through an opening in the bottom of the vessel. He then proceeded to show that a maximum in this dimensionless leakage time exists for certain vessel geometries as a function of the elevation (h_2) of the drainage hole.

This present work seeks to extend the analysis of side drainage from process vessels to several new geometries and to establish in which of these new geometries maximum drainage times can be found. Thus, equations are developed for side drainage from vessels of both cone and parboloid shapes, and with the vessel tip pointing up or down in either case, followed by a similar analysis for vertical ellipsoids.

2. Theory

The theory underlying the material balance and flow equations describing liquid efflux through a hole in a vented tank and solely under the influence of gravity can be found in many sources [1–6]. The primary assumptions invoked include turbulent flow through the hole and usage of the orifice equation to compensate for friction losses. Thus, liquid efflux from a vessel through an aperture with a cross-sectional area of A_o and located at an elevation of h_2 above the bottom of the vessel is described by the following nonlinear differential equation:

$$A\frac{\mathrm{d}h}{\mathrm{d}t} = -C_oA_o\sqrt{2g(h-h_2)}, \tag{1}$$

where h is the variable elevation of the liquid level above the bottom of the vessel, A is the area of the liquid level at that elevation and, depending upon the vessel geometry, generally a variable function of h, A_o is the area of the opening for liquid discharge, and C_o is an orifice discharge coefficient (a typically assumed value for which is 0.61).

Mathematically then, the problem of finding an explicit expression for the time (t) required for the liquid level in the vessel to drain from some initial elevation h_1 to some final elevation h_2 consists of integrating Eq. (1) between the appropriate limits. This is an extremely easy task in the case of process vessels for which A remains constant, irrespective of the liquid level — vertical circular or elliptical cylinders, parallelepipeds, etc. [8]. More complicated expressions for the drainage time result in the case of geometric shapes with variable horizontal areas, e.g., horizontal cylinders, paraboloids, spheres, etc. Thus, for example, in the case of a vertical cone (Fig. 1) with its tip pointing down, for which the cross-sectional area of the liquid level at any elevation h is $A = \pi h^2 \tan^2\theta$, the time for complete drainage

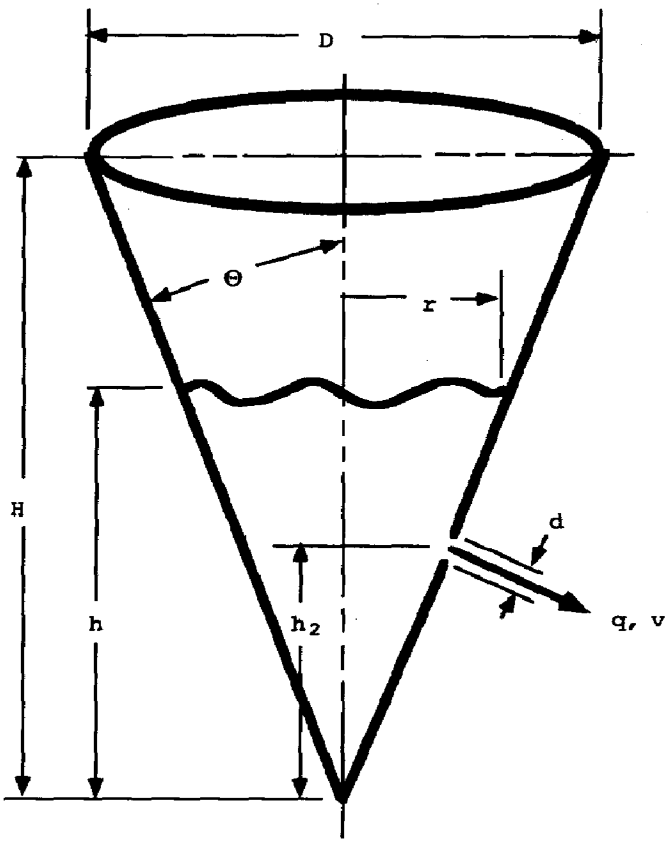

Fig. 1. Sketch of a conical process vessel with the cone tip pointing down and a puncture hole in its side.

from h_1 to h_2 is given by

$$t = \frac{2\pi \tan^2 \theta}{15 C_o A_o \sqrt{2g}} [3h_1^2 + 4h_1 h_2 + 8h_2^2] \sqrt{h_1 - h_2}, \tag{2}$$

where θ is the angle formed by the cone with its vertical axis. From Foster [2], the time (t_0) required for complete drainage of an initially full conical tank ($h_1 = H$, tip down) from a hole located at the tank bottom ($h_2 = 0$) is

$$t_0 = \frac{2\pi \tan^2 \theta}{5 C_o A_o \sqrt{2g}} H^{5/2}. \tag{3}$$

Consequently, the dimensionless drainage time ($\tau = t/t_0$) in this case is given by

$$\tau = \left[\frac{3x_1^2 + 4x_1 x_2 + 8x_2^2}{3} \right] \sqrt{x_1 - x_2}, \tag{4}$$

where x_1 and x_2 are the dimensionless elevations of the initial liquid level (h_1/H) and of the drainage hole (h_2/H), respectively.

3. Maximum drainage times

Crowl [5] first discovered that, for side drainage of initially full process vessels of certain geometric shapes, there can occur a maximum drainage or leakage time, which is a function of the elevation of the leakage hole and which can actually be greater than the time required to drain completely the full vessel through an identical hole at the bottom of the vessel. For example, he found that a maximum leakage time for a full spherical vessel occurs when the elevation of the drainage hole is equal to one fourth the sphere diameter; this maximum leakage time is 30% greater than that required for complete drainage through a hole of the same size located at the vessel bottom.

Similarly, Crowl showed via numerical methods that a maximum drainage time occurs for full horizontal circular cylindrical vessels when the elevation of the drainage hole above the vessel bottom is equal to 0.17 of the vessel diameter. Specifically, he reports that the maximum drainage time for this geometry is 14.8% greater than that for complete drainage from the vessel bottom. With the usage of elliptic integrals, Sommerfeld and Stallybrass [7] confirmed Crowl's results for horizontal circular cylindrical vessels. No such maximum is found in the case of drainage of vertical cylindrical vessels.

4. Conical vessels

A plot of Eq. (4) for the dimensionless drainage time τ of an initially full cone ($x_1 = 1$) with its tip down, as a function of the dimensionless elevation of the drainage hole (x_2), is provided in Fig. 2. This graph clearly illustrates the presence of a maximum in the dimensionless drainage time; this maximum may be found analytically by substituting $x_1 = 1$ into Eq. (4), differentiating with respect to x_2 and then setting the result equal to zero. These operations yield a value of $x_2 = (1 + \sqrt{3})/4 = 0.683$ for the location of the side drainage hole corresponding to the maximum drainage time. The value of this latter quantity in turn is equal to 1.776, again with $x_1 = 1$, which means that a full conical vessel will continue to drain for about 78% more time from a hole located at this elevation in the side of the vessel, in comparison with a hole at the vessel bottom.

For comparison purposes, side drainage of conical vessels with the cone tip pointing up, as shown in Fig. 3, was also investigated. Integration of the dynamic material balance equation for this particular configuration then results in the following expression for the drainage time:

$$t = \frac{2\pi \tan^2 \theta}{C_o A_o \sqrt{2g}} \left\{ H^2 - \frac{2H(h_1 + 2h_2)}{3} + \frac{3h_1^2 + 4h_1 h_2 + 8h_2^2}{15} \right\} \sqrt{h_1 - h_2}, \tag{5}$$

where H is the total height of the conical vessel. Similarly, the time (t_0) required for the special case of complete drainage of a full inverted cone (cone tip up, $h_1 = $ H,

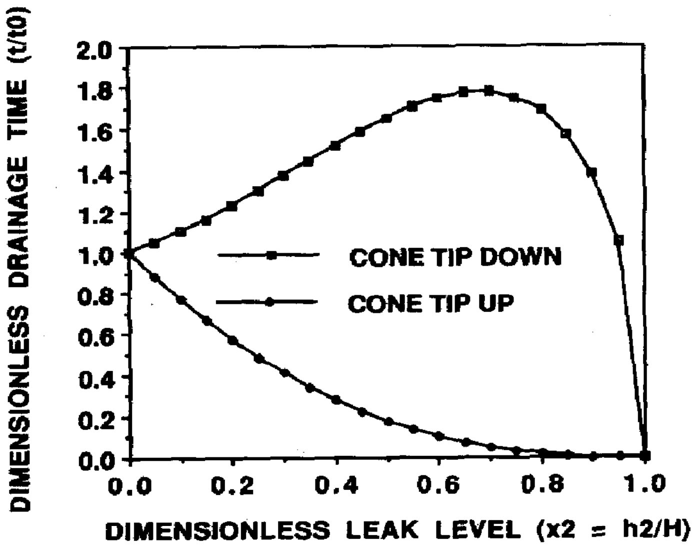

Fig. 2. Dimensionless side drainage times for initially full and vented conical vessels with the cone tip pointing down or up.

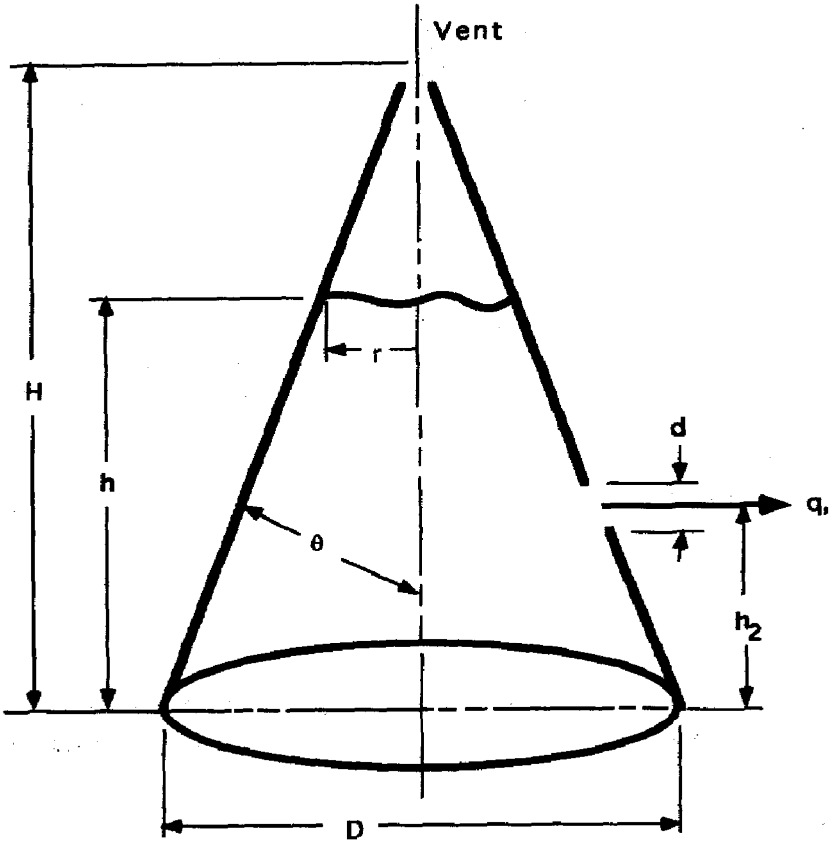

Fig. 3. Sketch of a conical process vessel with the cone tip pointing up and a puncture hole in its side.

$h_2 = 0$) vented to the atmosphere is given by

$$t_0 = \frac{16\pi \tan^2 \theta}{15 C_o A_o \sqrt{2g}} H^{5/2}. \tag{6}$$

To our knowledge, the above results for an inverted cone have never appeared in the technical literature. Interestingly enough, Eq. (6) states that the time required to drain completely a full inverted conical tank through a bottom hole is $\frac{8}{3}$ times the time required for complete drainage of the same full conical tank [but with the tip down, compare with Eq. (3)], all other parameters (C_o, A_o, H, θ) remaining the same.

Dividing Eq. (5) by Eq. (6) yields the following expression for the dimensionless drainage time for a conical tank with its tip up:

$$\tau = \left(\frac{15}{8}\right)\left\{1 - \frac{2(x_1 + 2x_2)}{3} + \frac{3x_1^2 + 4x_1x_2 + 8x_2^2}{15}\right\}\sqrt{x_1 - x_2}, \tag{7}$$

where the dimensionless heights x_1 and x_2 have the same significance as in the preceding case. This equation for the case of an initially full vessel ($x_1 = 1$) being drained down to the elevation of the side drainage hole is also depicted in Fig. 2, again with the dimensionless drainage time plotted versus the dimensionless elevation of the puncture hole. There is no maximum time exhibited in this curve for a conical vessel with its tip pointing up. Indeed, differentiation of Eq. (7) with respect to x_2 does not yield any intermediate stationary point, confirming the absence of a maximum in this particular vessel configuration.

5. Paraboloids

The specific vessel configuration considered here is a vertical paraboloid with a circular cross section in the horizontal plane. Thus, the area (A) of this cross section at any elevation h of the liquid level for a paraboloid with its tip pointing down (see Fig. 4) is equal to $\pi a^2 h/c$, where a is the radius of the paraboloid at its top and c is its total height. Insertion of this expression for the variable cross-sectional area into Eq. (1) followed by integration then yields the following result for the drainage time between h_1 and h_2:

$$t = \frac{2\pi a^2}{3 C_o A_o c \sqrt{2g}} (h_1 + 2h_2)\sqrt{h_1 - h_2}. \tag{8}$$

The complete efflux time requirement for an initially full paraboloid of circular cross section and draining through a hole in the bottom tip of the vessel has previously [8] been shown to be

$$t_0 = \frac{2\pi a^2}{3 C_o A_o \sqrt{2g}} \sqrt{c}. \tag{9}$$

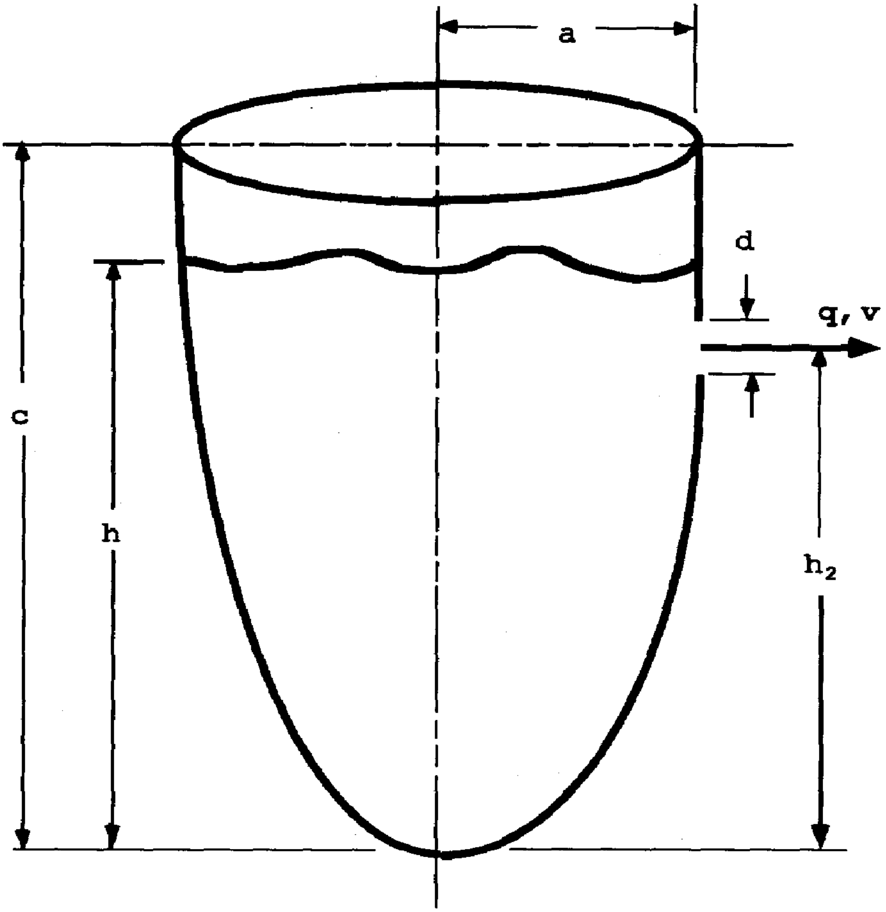

Fig. 4. Sketch of a paraboloid process vessel with its tip pointing down and a puncture hole in its side.

And thus the dimensionless drainage time for this particular vessel configuration, as given by dividing Eq. (8) by Eq. (9), is expressed by

$$\tau = (x_1 + 2x_2)\sqrt{x_1 - x_2}, \tag{10}$$

where $x_1 = h_1/c$ and $x_2 = h_2/c$. Differentiation of Eq. (10) with respect to x_2 (after setting $x_1 = 1$) in this case leads to the occurrence of a stationary point at $x_2 = 0.5$. The value of the function (maximum dimensionless drainage time) at this stationary point is then computed from Eq. (10) as equal to $\sqrt{2} = 1.414$.

Determination of the leakage time through a side hole in the case of a circular paraboloid with its tip pointing up, as depicted in Fig. 5, merely requires changing the expression for the area of the horizontal circular cross section to $A = \pi a^2(c - h)/c$, before substitution into Eq. (1) and integration. The resulting expression for the leakage time in this case is

$$t = \frac{2\pi a^2}{3C_o A_o c\sqrt{2g}}[3c - (h_1 + 2h_2)]\sqrt{h_1 - h_2}. \tag{11}$$

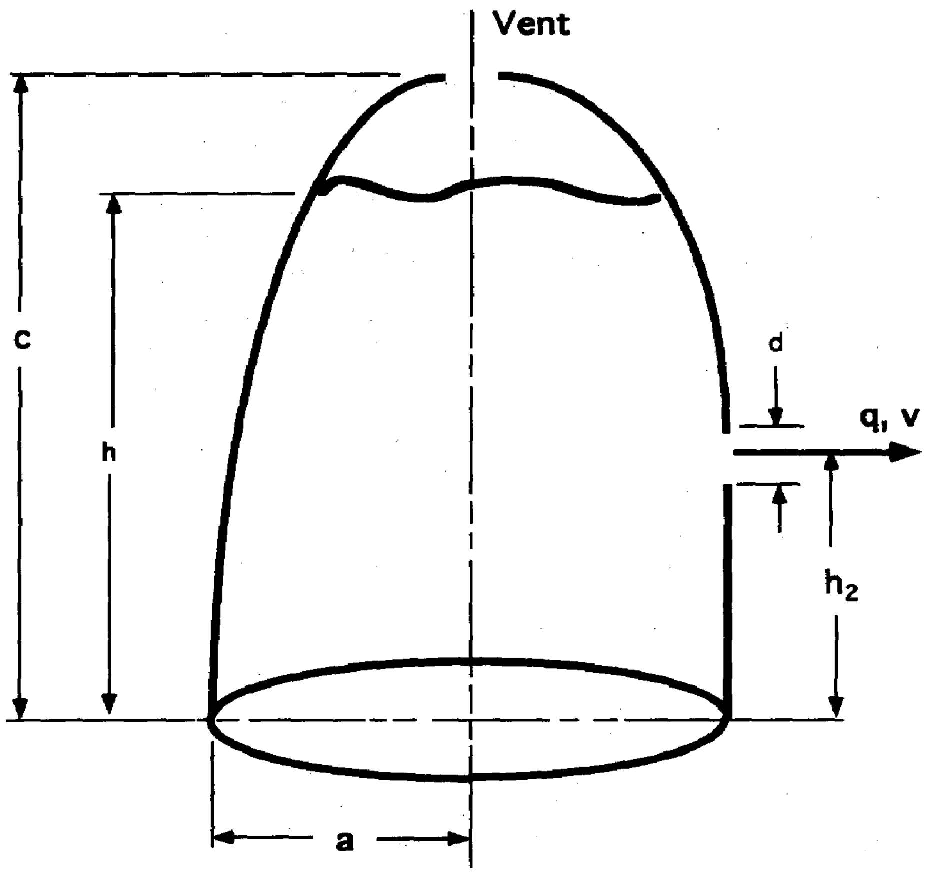

Fig. 5. Sketch of a paraboloid process vessel with its tip pointing up and a puncture hole in its side.

Similarly, the time requirement for complete drainage of such a paraboloid, initially full and through a leakage hole at the bottom, is given by setting $h_1 = c$ and $h_2 = 0$ in Eq. (11):

$$t_0 = \frac{4\pi a^2}{3C_o A_o \sqrt{2g}} \sqrt{c}. \tag{12}$$

In this case, comparison of Eq. (12) with Eq. (9) for complete drainage of a vertical circular paraboloid with its tip down leads to the interesting conclusion that the time for complete drainage of such a paraboloid with its tip up is exactly twice that required for one with its tip down. Lastly, the dimensionless drainage time τ in this case of a vertical circular paraboloid with its tip up, as given by dividing Eq. (11) by Eq. (12), is as follows:

$$\tau = \tfrac{1}{2}\,[3 - (x_1 + 2x_2)]\sqrt{x_1 - x_2} \tag{13}$$

again with $x_1 = h_1/c$ and $x_2 = h_2/c$. Here, as also in the case of a conical vessel with its tip up, differentiation of τ from Eq. (13) with respect to x_2 after

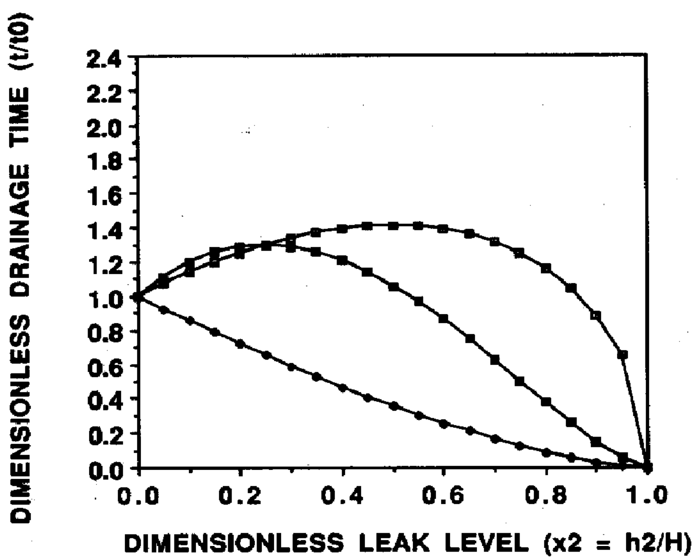

Fig. 6. Dimensionless side drainage times for initially full and vented paraboloids with the tip pointing down (□) or up (◆), and for circular ellipsoids (■) (vertical or horizontal).

substitution of $x_1 = 1$ does not lead to any intermediate stationary point. Thus, there exists no maximum leakage time either for a circular paraboloid with its tip up. The behavior of the dimensionless leakage times for vertical circular paraboloids with tips down and up, as given by Eqs. (10) and (13), respectively, is illustrated in Fig. 6.

6. Ellipsoids

We consider first a vertical ellipsoidal vessel with a circular cross section in the horizontal plane; such a configuration is shown in Fig. 7. Specifically, let the maximum radius of this cross section be equal to R and the overall height of this vessel H. The circular cross-sectional area at any elevation h of the liquid level is then given by

$$A = \frac{4\pi R^2}{H^2}(hH - h^2). \tag{14}$$

The drainage time for this vessel configuration, resulting from integration of Eq. (1) after insertion of Eq. (14) therein, is determined by the following expression:

$$t = \frac{8\pi R^2}{3C_o A_o H^2 \sqrt{2g}}\left[H(h_1 + 2h_2) - \frac{3h_1^2 + 4h_1 h_2 + 8h_2^2}{5}\right]\sqrt{h_1 - h_2}. \tag{15}$$

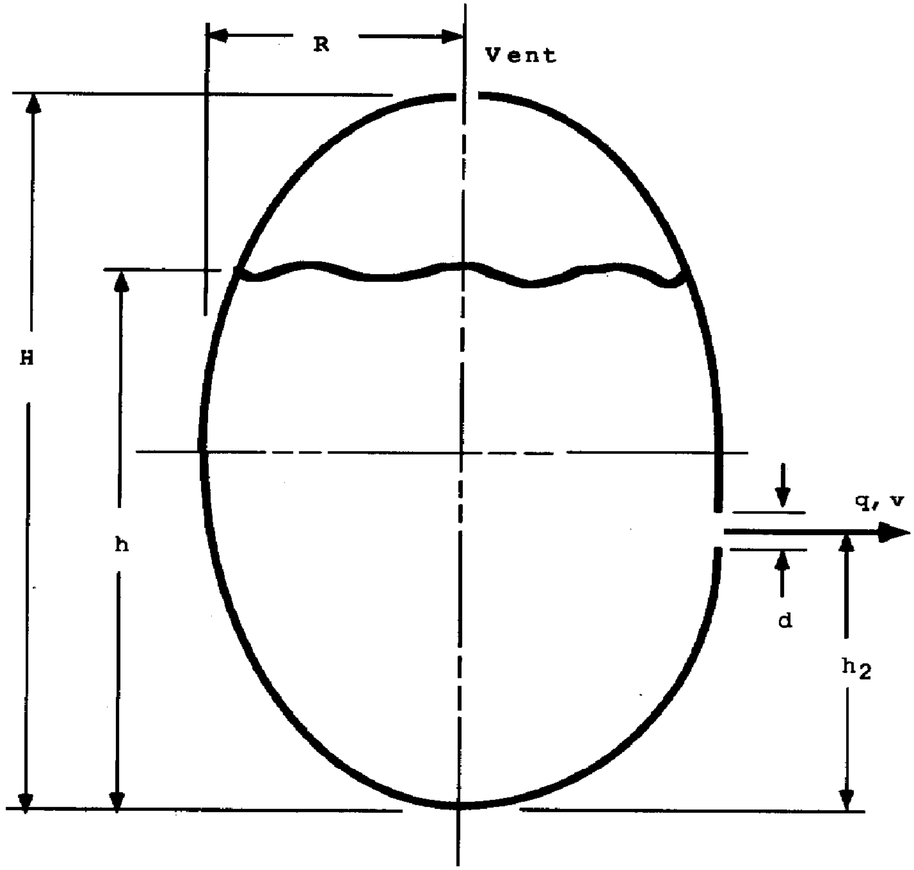

Fig. 7. Sketch of a vertical ellipsoidal process vessel with a circular cross section in the horizontal plane and a puncture hole in its side.

The complete efflux time for such an ellipsoidal vessel, initially full ($h_1 = H$) and draining through a hole at the bottom ($h_2 = 0$), is expressed as follows:

$$t_0 = \frac{16\pi R^2}{15 C_o A_o \sqrt{2g}} \sqrt{H}. \tag{16}$$

Equation (16) reduces to Foster's earlier expression [2] for complete drainage of a sphere when $H = 2R$. Dividing Eq. (15) by Eq. (16) then yields the dimensionless time τ for side drainage of a vertical ellipsoidal vessel with a circular horizontal cross section:

$$\tau = \tfrac{1}{2}\left[(5x_1 + 10x_2) - (3x_1^2 + 4x_1 x_2 + 8x_2^2)\right]\sqrt{x_1 - x_2}. \tag{17}$$

In this case, through usage of the calculus, a maximum in τ is found at $x_2 = 0.25$ when $x_1 = 1$; that is, the full vessel will continue to drain for the longest time when the leak occurs quarter of the way up the vessel height, which is exactly the same result originally obtained by Crowl [5] for a sphere. Specifically, this

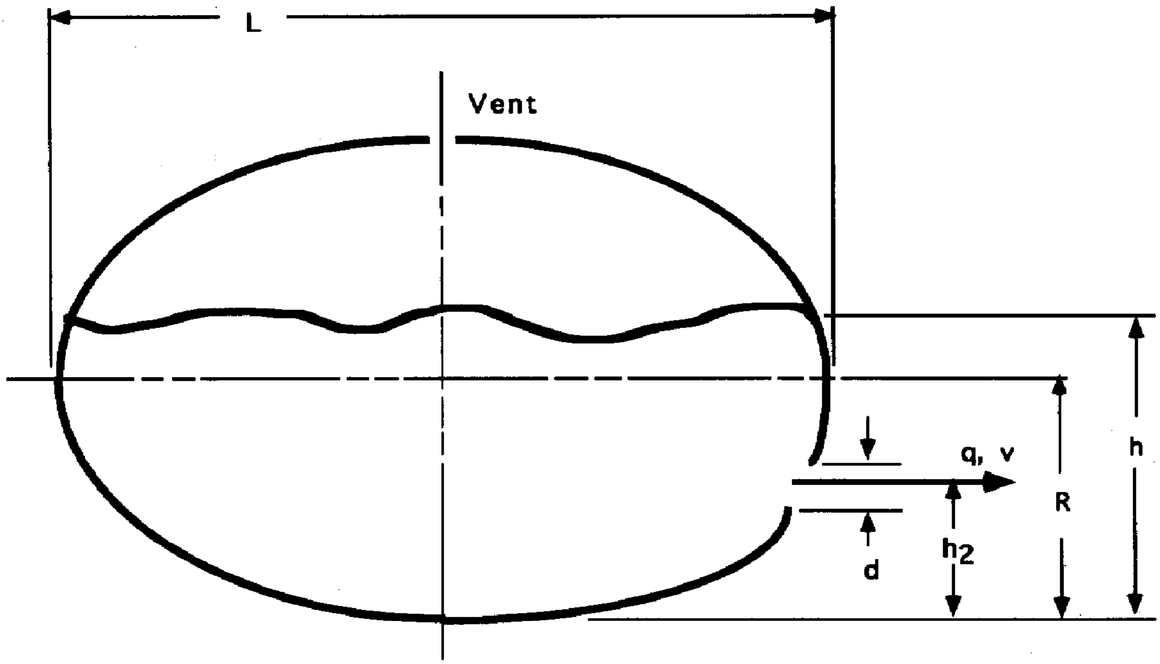

Fig. 8. Sketch of a horizontal ellipsoidal process vessel with a circular cross section in the vertical plane and a puncture hole in its side.

maximum value of the dimensionless drainage time τ for a full vertical ellipsoidal vessel is equal to $3\sqrt{3}/4 = 1.299$ — again, the same value as for a sphere.

Lastly, we consider a horizontal ellipsoidal vessel with a circular cross section in the vertical plane (see Fig. 8). In this case, we let the maximum radius of this cross section again be denoted by R and the overall length of this vessel in the horizontal plane by L. The elliptical cross-sectional area formed by the horizontal liquid surface at any liquid level h above the bottom of the vessel in this case is [9]

$$A = \frac{\pi L}{2R}[2hR - h^2] \tag{18}$$

and the drainage time is in turn given as

$$t = \frac{\pi L}{3C_o A_o R\sqrt{2g}}\left[2R(h_1 + 2h_2) - \frac{3h_1^2 + 4h_1 h_2 + 8h_2^2}{5}\right]\sqrt{h_1 - h_2}, \tag{19}$$

which is identical in form to Eq. (15) for a vertical circular ellipsoid. Similarly, when $h_1 = 2R$ and $h_2 = 0$, Eq. (19) reduces to the following expression:

$$t_o = \frac{8\pi LR}{15C_o A_o\sqrt{2g}}\sqrt{2R}. \tag{20}$$

In this case of a horizontal circular ellipsoid, the expression for the dimensionless drainage time τ [after dividing Eq. (19) by Eq. (20)] yields a result identical to Eq. (17) for a vertical circular ellipsoid, but with $x_1 = h_1/2R$ and $x_2 = h_2/2R$ in the horizontal case. Thus, the resulting values of the maximum dimensionless leakage time and the corresponding dimensionless elevation of the leak will be the same in both of the ellipsoidal cases, namely, $\tau = 1.299$ and $x_2 = 0.25$, respectively. It will be recalled that these are the same values as originally reported by Crowl [5] for a sphere. A plot of the behavior of this dimensionless leakage time for ellipsoids is also included in Fig. 6.

7. Discussion

Some discussion of the reasons for the appearance of maximum leakage times in certain geometric vessel configurations, but not in others, may be in order here. By way of convenience, maximum dimensionless leakage times and the corresponding dimensionless elevations of the leaks for all of the known vessel configurations examined to date are summarized in Table 1. As Crowl [5] first observed, there is no maximum leakage time in the case of a vertical circular cylindrical vessel — a vessel with a constant cross-sectional area formed by the liquid level therein. Thus, one would not anticipate the existence of maximum leakage times in the cases of vertical elliptical cylinders or parallelepipeds either — vessel shapes of constant cross-sectional area for which drainage time equations have also been developed earlier [8]. Also, as shown in this present work, no intermediate stationary points (maxima) are observed for vessel configurations wherein the liquid cross-sectional area is increasing with decreasing elevation of the liquid level, namely, conical vessels and paraboloids with their tips pointing up.

Table 1
Maximum dimensionless drainage times and corresponding dimensionless locations of the drainage hole in the vessel side for various geometrical vessel configurations

Vessel configuration	Maximum drainage time (τ_{max})	Elevation of drainage hole (x_2)	Reference(s)
Sphere	1.299	0.25	[5]
Vertical cylinder	None	—	[5]
Horizontal cylinder	1.161	0.17	[5,7]
Cone with tip down	1.776	0.683	This work
Cone with tip up	None	—	This work
Paraboloid with tip down	1.414	0.50	This work
Paraboloid with tip up	None	—	This work
Vertical ellipsoid	1.299	0.25	This work
Horizontal ellipsoid	1.299	0.25	This work

Hence, as Table 1 indicates, it is only in vessels wherein the cross-sectional area decreases with decreasing elevation of the liquid level that maxima in the dimensionless leakage time are found. In such geometric configurations — spheres, ellipsoids, horizontal cylinders, and cones and paraboloids with their tips pointing down — the volume of fluid remaining to be drained is decreasing as the liquid level falls. Thus, as the driving force (hydrostatic head) for drainage is dropping, the amount of fluid remaining to be drained under this reduced head is also decreasing. These two effects tend to counteract each other and result in an intermediate maximum drainage time. Again as indicated in Table 1, this maximum drainage time, as normalized to the time required to drain the full vessel from its bottom, never achieves the value of two, while the corresponding dimensionless elevation of the drainage hole varies from a value of 0.17 for a horizontal cylinder [5, 7] up to 0.683 for a conical vessel with its tip pointing down. It was noted earlier that the values — both drainage times and hole elevations — are the same for spheres and ellipsoids, and thus the former may be considered as merely a special case of the latter. Lastly, one would also expect to find maximum leakage times exhibited at intermediate elevations of the drainage hole by any other process vessels whose geometric configuration is such that the cross-sectional area formed by the liquid level decreases as the elevation of the leakage hole falls.

Nomenclature

a	length of major axis of an ellipse, m
A	cross-sectional area of the liquid surface in the vessel at any time, m^2
A_o	cross-sectional area of the opening for liquid flow out of the vessel, m^2
b	length of minor axis of an ellipse, m
c	height of a paraboloid, m
C	length of chord formed by liquid level in a vessel, m
C_o	orifice discharge coefficient
d	diameter of the opening for liquid flow out of the vessel, m
D	diameter of a vessel, m
g	acceleration due to gravity, m/s^2
h	elevation of liquid level in vessel at any time, m
h_1	initial elevation ($t = 0$) of liquid level in the vessel, m
h_2	elevation of opening above the bottom of the vessel, m
H	overall height of a vessel, m
L	length of a horizontal vessel, m
q	liquid volumetric flow rate out of the vessel, m^3/s
r	variable radius of circular cross section formed by the liquid level, m
R	radius of a vessel ($= D/2$), m
t	time, s
t_0	time for complete drainage of a full vessel through a hole located at the bottom, s
v	liquid velocity, m/s

V liquid volume in the vessel at any time, m^3
x dimensionless elevation of liquid level ($= h/D$)
x_1 dimensionless elevation of initial liquid level ($= h_1/D$)
x_2 dimensionless elevation of opening in vessel ($= h_2/D$ or h_2/H)

Greek letters

θ angle formed by a cone with the vertical axis
π number pi (3.14159...)
ρ liquid density, kg/m^3
τ dimensionless drainage time ($= t/t_0$)

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